M11/5/MATHL/HP3/ENG/TZ0/SG

22117209

MATHEMATICS
HIGHER LEVEL
PAPER 3 – SETS, RELATIONS AND GROUPS

Monday 9 May 2011 (morning)

1 hour

INSTRUCTIONS TO CANDIDATES

- Do not open this examination paper until instructed to do so.
- Answer all the questions.
- Unless otherwise stated in the question, all numerical answers must be given exactly or correct to three significant figures.

Please start each question on a new page. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. In particular, solutions found from a graphic display calculator should be supported by suitable working, e.g. if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

1. *[Maximum mark: 18]*

The binary operator multiplication modulo 14, denoted by $*$, is defined on the set $S = \{2, 4, 6, 8, 10, 12\}$.

(a) Copy and complete the following operation table.

$*$	**2**	**4**	**6**	**8**	**10**	**12**
2						
4	8	2	10	4	12	6
6						
8						
10	6	12	4	10	2	8
12						

[4 marks]

(b) (i) Show that $\{S, *\}$ is a group.

(ii) Find the order of each element of $\{S, *\}$.

(iii) Hence show that $\{S, *\}$ is cyclic and find all the generators. *[11 marks]*

(c) The set T is defined by $\{x * x : x \in S\}$. Show that $\{T, *\}$ is a subgroup of $\{S, *\}$. *[3 marks]*

2. *[Maximum mark: 7]*

The universal set contains all the positive integers less than 30. The set A contains all prime numbers less than 30 and the set B contains all positive integers of the form $3 + 5n$ $(n \in \mathbb{N})$ that are less than 30. Determine the elements of

(a) $A \setminus B$; *[4 marks]*

(b) $A \Delta B$. *[3 marks]*

3. *[Maximum mark: 10]*

The relation R is defined for $a, b \in \mathbb{Z}^+$ such that aRb if and only if $a^2 - b^2$ is divisible by 5.

(a) Show that R is an equivalence relation. *[6 marks]*

(b) Identify the three equivalence classes. *[4 marks]*

4. *[Maximum mark: 11]*

The function $f : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+ \times \mathbb{R}^+$ is defined by $f(x, y) = \left(xy^2, \dfrac{x}{y}\right)$.

Show that f is a bijection.

5. *[Maximum mark: 14]*

(a) Given that p, q and r are elements of a group, prove the left-cancellation rule, *i.e.* $pq = pr \Rightarrow q = r$.

Your solution should indicate which group axiom is used at each stage of the proof. *[4 marks]*

(b) Consider the group G, of order 4, which has distinct elements a, b and c and the identity element e.

(i) Giving a reason in each case, explain why ab cannot equal a or b.

(ii) Given that c is self inverse, determine the two possible Cayley tables for G.

(iii) Determine which one of the groups defined by your two Cayley tables is isomorphic to the group defined by the set $\{1, -1, \mathrm{i}, -\mathrm{i}\}$ under multiplication of complex numbers. Your solution should include a correspondence between a, b, c, e and $1, -1, \mathrm{i}, -\mathrm{i}$. *[10 marks]*